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ON l^p-COMPLEMENTED COPIES IN ORLICZ SPACES†

BY

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ABSTRACT

Let $1 < \alpha \leqq \beta < \infty$ and F be an arbitrary closed subset of the interval $[\alpha, \beta]$. An Orlicz sequence space l^ϕ (resp. an Orlicz function space $L^\phi(\mu)$) with associated indices α and β is found in such a way that the set of values p for which the l^p-space is isomorphic to a complemented subspace of l^ϕ (resp. $L^\phi(\mu)$) is precisely the given set F (resp. $F \cup \{2\}$). Also, a recent result of Hernández and Peirats [1] is extended showing that, even for the case in which the indices satisfy $\alpha_\phi^\infty < 2 < \beta_\phi^\infty$, there exist minimal Orlicz function spaces $L^\phi(\mu)$ with no complemented copy of l^p for any $p \neq 2$.

0. Introduction

The study of symmetric structure of the Orlicz spaces has been carried out mainly and among others by J. Lindenstrauss and L. Tzafriri ([5], [6], [7]), N. Kalton ([4]) and N. Nielsen ([11]) (see also [3], [8], [9]) offering several important and deep results. To take a sample, let us mention that the class of the minimal Orlicz sequence spaces l^ϕ studied by J. Lindenstrauss and L. Tzafriri provided examples of Banach spaces containing isomorphic copies of l^p-spaces for uncountable many p's and, at the same time, no *complemented* copy of any l^q-space (see [1] for the version of this theorem in the context of Orlicz function spaces).

In general, the problem of determining exactly which spaces l^ψ (in particular l^p-spaces) are isomorphic to a complemented subspace of an Orlicz sequence space l^ϕ does not have yet a complete solution, and full characterization

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remains to be found. However, some useful necessary or sufficient conditions are known. Thus, J. Lindenstrauss and L. Tzafriri, in [6], introduced the notion of strongly non-equivalent function to the set $E_{\phi,1}$, proving that a space l^{ψ} is not isomorphic to any complemented subspace of an Orlicz sequence space l^{ϕ} if the function ψ is strongly non-equivalent to $E_{\phi,1}$.

This paper deals with the l^p-complemented copies ($1 \leqq p < \infty$) in Orlicz spaces: One of its goals is to extend to the context of Orlicz function spaces $L^{\phi}(\mu)$ the above-mentioned result on strongly non-equivalent functions in sequence spaces l^{ϕ} ([6], [7]). Another purpose of this paper is to study the following *inverse problem*: Given an arbitrary set F of real numbers $p \geqq 1$, find Orlicz (sequence and function) spaces $L^{\phi}(\mu)$ such that the set of values p for which the l^p-space can be complementably embedded into $L^{\phi}(\mu)$ is exactly the prefixed set F.

In Section II we answer this problem for closed sets F and Orlicz sequence spaces. Thus, the main result (Theorem 2) asserts that given $1 < \alpha \leqq \beta < \infty$ and a closed subset F of the interval $[\alpha, \beta]$, there exists always an Orlicz sequence space l^{ϕ} with indices $\alpha_{\phi} = \alpha$ and $\beta_{\phi} = \beta$ which contains a complemented copy of l^p if and only if $p \in F$. The proof of this result makes use of the method of constructing Orlicz functions associated with sequences of 0's and 1's, which was developed by J. Lindenstrauss and L. Tzafriri in ([6], [7], [8]).

Section III is devoted to Orlicz function spaces: The first part introduces the concept of strongly non-equivalent function to $E^{\infty}_{\phi,1}$, studying its connection with weighted Orlicz sequence spaces $l^{\phi}(\mu)$. As a consequence, this allows us to give a criterion to insure that reflexive Orlicz function spaces $L^{\phi}(\mu)$ contain no complemented copies of l^p ($p \neq 2$): The function t^p must be strongly non-equivalent to $E^{\infty}_{\phi,1}$. This fact constitutes a partial extension of ([6] Theorem 2.2) to Orlicz function spaces. Also, in this context we solve the above-mentioned *inverse problem* for closed sets $F \cup \{2\}$ (Theorem 7).

Finally, some applications to the class of *minimal* Orlicz function spaces are given. In particular, Corollary 10 answers in the affirmative a question in [1] (Remark, page 360), showing that the main result in [1] is also true for the case $2 \in (\alpha^{\infty}_{\phi}, \beta^{\infty}_{\phi})$. Thus, there exist reflexive Orlicz function spaces $L^{\phi}(\mu)$ with arbitrary indices without complemented copies of l^p for any $p \neq 2$.

I. Preliminaries

Let us start with some notations and definitions. Given a positive measure space (Ω, Σ, μ) and an Orlicz function ϕ (i.e., a continuous convex nondecreasing function defined for $x \geqq 0$ so that $\phi(0) = 0$ and $\phi(1) = 1$), the *Orlicz function space* $L^\phi(\mu)$ is defined as the set of equivalence classes of μ-measurable scalar functions of (Ω, Σ, μ) such that

$$m_r(f) = \int_\Omega \phi\left(\frac{|f|}{r}\right) d\mu < \infty, \qquad \text{for some } r > 0.$$

The space $L^\phi(\mu)$ endowed with the Luxemburg norm $\|f\| = \inf\{r > 0 : m_r(f) \leqq 1\}$ is a Banach space.

Similarly the *Orlicz sequence space* l^ϕ consists of all those sequences $x = (x_n)$ of scalars for which there is an $r > 0$ with

$$m_r(x) = \sum_{n=1}^{\infty} \left(\frac{|x_n|}{r}\right) < \infty.$$

Recall that a function ϕ satisfies the Δ_2-condition at 0 (resp. at ∞) if there exist constants $M > 0$ and $s_0 > 0$ with $\phi(s_0) > 0$ such that $\phi(2s) \leqq M\phi(s)$ if $0 < s \leqq s_0$ (resp. $s \geqq s_0$). If ϕ satisfies the Δ_2-condition at 0 then the sequence of unit vectors (e_n) is a symmetric basis of l^ϕ.

We assume that the Orlicz function ϕ satisfies the Δ_2-condition at ∞ and at 0, so the associated indices verify $1 \leqq \alpha_\phi^\infty \leqq \beta_\phi^\infty < \infty$ and $1 \leqq \alpha_\phi \leqq \beta_\phi < \infty$ (cf. [8], [9]). We shall make use of the following compact subsets related to ϕ in the space $C(0, 1)$:

$$E_{\phi,s} = \overline{\left\{\frac{\phi(rt)}{\phi(r)} : r \leqq s\right\}}; \qquad E_\phi = \bigcap_{s>0} E_{\phi,s}$$

$$E_{\phi,s}^\infty = \overline{\left\{\frac{\phi(rt)}{\phi(r)} : r \geqq s\right\}}; \qquad E_\phi^\infty = \bigcap_{s>0} E_{\phi,s}^\infty$$

$$C_{\phi,s} = \overline{\text{conv}}\, E_{\phi,s}; \qquad C_\phi = \overline{\text{conv}}\, E_\phi$$

$$C_{\phi,s}^\infty = \overline{\text{conv}}\, E_{\phi,s}^\infty; \qquad C_\phi^\infty = \overline{\text{conv}}\, E_\phi^\infty$$

for every $s > 0$ ([5–8]). For a detailed study of Orlicz spaces and their structure we refer to ([10], [8], [9]).

It was proved in [7] that for Orlicz sequence spaces l^ϕ the following statements are equivalent: (a) l^p is isomorphic to a subspace of l^ϕ; (b) $p \in [\alpha_\phi, \beta_\phi]$; (c) the function t^p is equivalent to some function in $C_{\phi,1}$. (Furthermore when t^p is in $E_{\phi,1}$, the space l^p is isomorphic to a complemented subspace of l^ϕ.) For Orlicz function spaces $L^\phi(\mu)$ the following holds: $p \in [\alpha_\phi^\infty, \beta_\phi^\infty]$ if and only if the unit vector basis of l^p is equivalent to a sequence of functions in $L^\phi(\mu)$ with mutually disjoint supports.

On the other hand, an Orlicz sequence space l^ϕ does not contain any complemented subspace isomorphic to l^p if the function t^p is strongly non-equivalent to $E_{\phi,1}$ ([6], [8] Theorem 4.b.5). In general, a function ψ is *strongly non-equivalent* to $E_{\phi,1}$ if for every constant $K \geqq 1$ there exists $m(K)$-points $t_i \in (0, \frac{1}{2})$ such that, if $K \to \infty$, $m(K) = o(K^\sigma)$ for every $\sigma > 0$, and for every $\lambda \in (0, 1)$ there exists at least one index i, $1 < i \leqq m(K)$ for which

$$\frac{\phi(\lambda t_i)}{\phi(\lambda)\psi(t_i)} \notin \left[\frac{1}{K}, K\right].$$

(For the ordinary equivalence — instead of "strongly" — the above result is not true; see N. Kalton [4].)

The class of the minimal Orlicz sequence spaces l^ϕ has been studied in ([6], [7], see also [8]): An Orlicz function ϕ is called *minimal* (at 0) if, for every function $\psi \in E_{\phi,1}$, we have that $E_{\psi,1} = E_{\phi,1}$. Examples 4.c.6 and 4.c.7 in ([8]) prove the existence of minimal Orlicz sequence spaces l^ϕ with arbitrary indices $(1 < \alpha_\phi \leqq \beta_\phi < \infty)$ containing no complemented subspaces isomorphic to l^p for any $p \geqq 1$. For Orlicz function spaces it has been proved in [1], by means of an extension of the notion of minimal function, that there exist reflexive Orlicz function spaces $L^\phi(\mu)$ with indices $1 < \alpha_\phi^\infty \leqq \beta_\phi^\infty \leqq 2$ (or $2 \leqq \alpha_\phi^\infty \leqq \beta_\phi^\infty < \infty$) containing no complemented subspaces isomorphic to l^p for any $p \neq 2$.

II. l^p-complemented copies in Orlicz sequence spaces l^ϕ

In this section we deal with the following problem: given a set F of real numbers $p \geqq 1$, find an Orlicz sequence space l^ϕ such that the set of values p for which the space l^p is isomorphic to a complemented subspace of l^ϕ is exactly the set F.

The next theorem establishes a first step towards the general solution for closed sets given in Theorems 2 and 3.

THEOREM 1. *Let F be a closed set of positive numbers with $1 \leqq \alpha = \inf F \leqq \sup F = \beta < \infty$. Then there exists an Orlicz sequence space l^ϕ with indices $\alpha_\phi = \alpha$* and *$\beta_\phi = \beta$ which contains complemented subspaces isomorphic to l^p if and only if $p \in F$. Furthermore t^p belongs to E_ϕ, up to equivalence, for each $p \in F$, and t^p is strongly non-equivalent to $E_{\phi,1}$ for each $p \notin F$.*

PROOF. Firstly, by the separability of F, we pick up a dense sequence (p_n) in F, verifying that every term of the range of (p_n) appears infinitely many times in the sequence.

Let us consider the following function f on $[0, +\infty)$ defined as follows:

$$f(t) = 0 \qquad \text{for } t \in [0, 1],$$

$$f(t) = f(n^2) + p_n(t - n^2) \qquad \text{for } t \in [n^2, (n+1)^2], \quad n \in \mathbf{N}.$$

It is clear that $|f(t_1) - f(t_2)| \leqq \beta |t_1 - t_2|$.

Now, let us define a function φ on $[0, 1]$ by $\varphi(0) = 0$ and

$$\varphi(t) = \exp\{-f(-\log t)\} \qquad \text{for } 0 < t \leqq 1.$$

This function φ is continuous but not necessarily convex on $[0, 1]$. However, if φ' denotes the right-derivative of φ we have

$$\alpha \leqq \frac{t\varphi'(t)}{\varphi(t)} = f'(-\log t) \leqq \beta$$

for every $t \in (0, 1)$. Thus, since $\alpha \geqq 1$ we get that $\varphi(t)/t$ is an increasing function, and hence φ is equivalent to the convex function

$$\phi(t) = \int_0^t \frac{\varphi(u)}{u}\, du \qquad \text{for } t \in [0, 1].$$

So ϕ satisfies the Δ_2-condition at 0.

Let $r_n = e^{-n^2}$, we consider the sequence of functions $(\varphi_n)_1^\infty \subset E_{\varphi,1}$ defined for $t \in [0, 1]$ by

$$\varphi_n(t) = \frac{\varphi(r_n t)}{\phi(r_n)} = \exp\{f(n^2) - f(n^2 - \log t)\}.$$

Then for $r_{n+1}/r_n \leqq t \leqq 1$, it is easily checked that $\varphi_n(t) = t^{p_n}$. Now, as $r_{n+1}/r_n \to 0$ and each value p_n of the range of (p_n) appears infinitely many times in the sequence (p_n), there exists a subsequence (φ_{n_k}) of (φ_n), which converges uniformly to t^{p_n} on $[0, 1]$. Therefore $t^{p_n} \in E_\varphi$ for all $n \in \mathbf{N}$ and hence $t^p \in E_\varphi$ for every $p \in F$. Thus t^p is equivalent at 0 to a function of E_ϕ for every $p \in F$, and, by (cf. [8] page 150), l^ϕ has a complemented copy of l^p. Moreover, it is easy to show that the indices of the function ϕ are exactly $\alpha_\phi = \alpha$ and $\beta_\phi = \beta$.

We pass now to prove that t^p is strongly non-equivalent to $E_{\phi,1}$ for each $p \notin F$, which implies, by Theorem 4.b.5 in [8], that l^ϕ does not contain any complemented subspace isomorphic to l^p. Given $p \notin F$, let $\varepsilon > 0$ so that

$(p-3\varepsilon, p+3\varepsilon) \cap F = \varnothing$. For each integer n put $m(n) = n^2$ and assume the existence of an integer k such that

$$(*) \qquad e^{-\varepsilon n} \leqq \frac{\varphi(\tau^k \tau^i)}{\varphi(\tau^k)\tau^{pi}} \leqq e^{\varepsilon n}$$

for $i = 1, 2, \ldots, n^2$ and $\tau = e^{-1}$. Let $1 \leqq j \leqq n^2 - n$ $(n > 1)$, by using the above inequality with $i = j$ and $i = j + n$, we obtain

$$e^{-2\varepsilon n}\tau^{pn} \leqq \frac{\varphi(\tau^{k+j+n})}{\varphi(\tau^{k+j})} \leqq e^{2\varepsilon n}\tau^{pn}$$

for $1 \leqq j \leqq n^2 - n$.

We consider now a particular value of j in each one of the following cases:

(a) $j = (n-1)^2 - k$ when $k < (n-1)^2$;

(b) $j = 1$ when $(m-1)^2 \leqq k < k + n < m^2$ for an integer $m \geqq n$;

(c) $j = m^2 - k$ when $(m-1)^2 \leqq k < m^2 \leqq (k+n)$ for an integer $m \geqq n$.

Then, it is easily checked that in each case

$$\frac{\varphi(\tau^{k+j+n})}{\varphi(\tau^{k+j})} = \tau^{qn}$$

for some $q = p_i \in F$. But, as $|q - p| \geqq 3\varepsilon$, the above equality implies

$$\frac{\varphi(\tau^{k+j+n})}{\varphi(\tau^{k+j})\tau^{pn}} \notin [e^{-\varepsilon n}, e^{\varepsilon n}],$$

which is a contradiction with $(*)$. So, we have obtained that for any integer n there exists $m(n) = n^2$ points in $(0, 1)$ such that for any integer k there is at least one index $i = 1, \ldots, n^2$ for which

$$\frac{\varphi(\tau^{i+k})}{\varphi(\tau^k)\tau^{pi}} \notin [e^{-\varepsilon n}, e^{\varepsilon n}].$$

As $m(n+1) = o(e^{\sigma\varepsilon n})$ for any $\sigma > 0$, this means, by the Δ_2-condition at 0, that l^p is strongly non-equivalent to $E_{\varphi,1}$, which ends the proof. q.e.d.

In the proof of the following result we will make use of the method of constructing Orlicz functions by sequences of 0's and 1's developed by J. Lindenstrauss and L. Tzafriri in ([6], [7], [8]).

Let us recall that, for fixed $0 < \tau < 1$ and $1 < p' < p''$, if $\rho = (\rho(n))_1^\infty$

denotes a sequence of digits with $\rho(n)$ equal to 0 or 1 for each $n \in \mathbf{N}$, then the Orlicz function φ_ρ associated with the sequence $\rho(n)$ ([8] page 161) verifies that

$$\varphi_\rho(\tau^k) = \tau^{p'k + (p''-p')\sum_{n=1}^k \rho(n)} \qquad \text{for } k \in \mathbf{N};$$

and its indices satisfy ([8] Proposition 4.c.4)

$$\alpha_{\varphi_\rho} = p' + (p'' - p')a_0 \qquad \text{and} \qquad \beta_{\varphi_\rho} = p' + (p'' - p')b_0,$$

where a_0 and b_0 are real numbers depending on the density of 1's of ρ, defined by

$$a_0 = \varliminf_{k\to\infty} \inf_n \frac{1}{k} \sum_{i=n+1}^{n+k} \rho(i),$$

$$b_0 = \varlimsup_{k\to\infty} \sup_n \frac{1}{k} \sum_{i=n+1}^{n+k} \rho(i).$$

THEOREM 2. *Let $1 < \alpha \leqq \beta < \infty$ and F be an arbitrary closed subset of the interval $[\alpha, \beta]$. Then there exists an Orlicz sequence space l^ϕ, with indices $\alpha_\phi = \alpha$ and $\beta_\phi = \beta$, which contains a complemented copy of l^p if and only if $p \in F$.*

PROOF. By the results of Lindenstrauss and Tzafriri ([7], [8] Examples 4.c.6, and 4.c.7) we can assume that $F \neq \emptyset$.

First, let us show that we can take numbers $1 < p' < p''$ and a sequence $\rho = (\rho(n))$ of 0's and 1's such that the associated Orlicz function φ_ρ has indices $\alpha_{\varphi_\rho} = \alpha < \beta_{\varphi_\rho} = \beta$. Indeed, for $0 < a < b < 1$ and $a/b < (\alpha - 1)/(\beta - 1)$ we construct a sequence of positive integers (n_j), with $n_j > 5$, such that

$$\sum_{j=1}^{\infty} \frac{1}{n_j} \leqq a \quad \text{and} \quad \prod_{j=1}^{\infty} \left(1 - \frac{1}{n_j}\right) \geqq b.$$

Proceeding as in ([8] page 165) we define two sequences of zeros and ones, as follows. Let $m_j = n_1 n_2 \cdots n_{j-1}$ ($m_1 = 1$) and A_j (resp. B_j) be the block of the first m_j digits of ρ (resp. η). Thus A_1 consists of the digit 1 and B_1 of the digit 0, while A_j and B_j are defined inductively by

$$A_{j+1} = \overbrace{A_j A_j \cdots A_j}^{(n_j - 1)\text{ times}} B_j, \qquad B_{j+1} = \overbrace{B_j B_j \cdots B_j}^{(n_j - 1)\text{ times}} A_j$$

It is clear that for a_j (resp. b_j) denoting the number of ones inside A_j (resp. B_j),

$$a_{j+1} = (n_j - 1)a_j + b_j; \qquad b_{j+1} = (n_j - 1)b_j + a_j$$

and, hence,

$$a_{j+1}-b_{j+1}=(n_j-2)(a_j-b_j)=\prod_{i=1}^{j}(n_i-2)\geqq\theta(m_{j+1})$$

where $\theta=\Pi_{j=1}^{\infty}(1-2/n_j)>0$. Finally, as $a_0\leqq a<b\leqq b_0$ we can find $p'<p''$ verifying

$$\alpha=p'+(p''-p')a_0=\alpha_{\varphi_p},$$

$$\beta=p'+(p''-p')b_0=\beta_{\varphi_p},$$

with

$$p'=1+\frac{(\alpha-1)b_0-(\beta-1)a_0}{b_0-a_0}>1$$

because

$$\frac{a_0}{b_0}\leqq\frac{a}{b}<\frac{\alpha-1}{\beta-1}.$$

We consider now, as in Theorem 1, a dense sequence $(p_{2n-1})_1^{\infty}$ in the set F, so that every element of the range appears infinitely many times. Let us define a continuous function φ on $[0, 1]$ as follows: $\varphi(0)=0$, $\varphi(1)=1$, and

$$\varphi(x)=\begin{cases}\varphi_p\left(\dfrac{x}{r_n}\right)\varphi(r_n) & \text{if } r_{n+1}\leqq x\leqq r_n \text{ and } n \text{ even}\\[2ex] \left(\dfrac{x}{r_n}\right)^{p_n}\varphi(r_n) & \text{if } r_{n+1}\leqq x\leqq r_n \text{ and } n \text{ odd}\end{cases}$$

where $r_n=e^{-m_n}$ for $n\in\mathbf{N}$ $(r_0=1)$ and φ_p is the above Orlicz function associated with p and $\varphi_p(1)=1$. This function φ is not necessarily convex, but as $\varphi(x)/x$ is an increasing function, the function φ is equivalent at 0 to the convex function ϕ defined by

$$\phi(t)=\int_0^t\frac{\varphi(u)}{u}\,du \qquad \text{for } t\in[0, 1].$$

Hence ϕ satisfies the Δ_2-condition at 0 too.

In the same way as in Theorem 1, by considering the sequence

$$(\varphi_n(t)) = \left(\frac{\varphi(r_n t)}{\varphi(r_n)}\right)$$

for n odd, it is proved that for each $p \in F$ the function $t^p \in E_\varphi$ and hence the sequence space l^ϕ has a complemented copy of l^p.

Now, in order to prove that l^ϕ does not contain any complemented subspace isomorphic to l^p if $p \notin F$, we will show that t^p is strongly non-equivalent to $E_{\phi,1}$, and then we apply ([6], [8] Theorem 4.b.5). Fix $p \notin F$, let $\varepsilon > 0$ be such that $(p - 3\varepsilon, p + 3\varepsilon) \cap F = \varnothing$. For each odd integer n, put $m(n) = 5m_{n+1}$ and assume the existence of an integer k such that

$$K_n^{-1}\tau^{pi} \leqq \frac{\varphi(\tau^{k+i})}{\varphi(\tau^k)} \leqq K_n \tau^{pi}$$

for $i = 1, 2, \ldots, 5m_{n+1}$ and where $K_n = e^{\delta m_n}$ for $\delta > 0$ and $\tau = e^{-1}$. Let $1 \leqq j \leqq 4m_{n+1}$; by the above inequality with $i = j$ and $i = j + m_n$, we obtain

$$(*) \qquad K_n^{-2}\tau^{pm_n} \leqq \frac{\varphi(\tau^{k+j+m_n})}{\varphi(\tau^{k+j})} \leqq K_n^2 \tau^{pm_n}$$

for $1 \leqq j \leqq 4m_{n+1}$.

We distinguish now the following five cases depending on the possible values of k:

(a) $k < m_n$,

(b) $m_{n'} \leqq k \leqq k + 4m_{n+1} < m_{n'+1}$ for some odd integer $n' \geqq n$,

(c) $m_{n'} \leqq k \leqq k + 4m_{n+1} < m_{n'+1}$ for some even integer $n' \geqq n$,

(d) $m_{n'} \leqq k < m_{n'+1} \leqq k + 4m_{n+1}$ for some even integer $n' \geqq n$,

(e) $m_{n'} \leqq k < m_{n'+1} \leqq k + 4m_{n+1}$ for some odd integer $n' \geqq n$.

For the cases (a), (b) and (d) let $j = m_n - k$, $j = 1$, and $j = m_{n'+1} - k$ ($\leqq 4m_{n+1}$) respectively. Then, for these cases, it is easily checked that

$$K_n^{-2}\tau^{pm_n} \leqq \frac{\varphi(\tau^{k+j+m_n})}{\varphi(\tau^{k+j})} = \tau^{qm_n} \leqq K_n^2 \tau^{pm_n}$$

for some $q = p_{2i-1} \in F$. From $|q - p| \geqq 3\varepsilon$ we deduce that $K_n > e^{\varepsilon m_n}$, so $\delta > \varepsilon$.

Case (c). As every block of ρ or η of length $3m_{n+1}$ contains one block equal to A_n and another block equal to B_n, there exist integers j_1, j_2 with $1 \leqq j_1$, $j_2 \leqq 3m_{n+1}$ for which

$$\frac{\varphi_\rho(\tau^{k+j_1+m_n-m_{n'}})}{\varphi_\rho(\tau^{k+j_1-m_{n'}})} = \varphi_\rho(\tau^{m_n})$$

and

$$\frac{\varphi_\rho(\tau^{k+j_2+m_n-m_{n'}})}{\varphi_\rho(\tau^{k+j_2-m_{n'}})}=\varphi_\eta(\tau^{m_n}).$$

Now, using ($\overset{*}{*}$) we obtain

$$K_n^{-4}\leqq\frac{\varphi_\rho(\tau^{m_n})}{\varphi_\eta(\tau^{m_n})}\leqq K_n^4.$$

On the other hand

$$\frac{\varphi_\rho(\tau^{m_n})}{\varphi_\eta(\tau^{m_n})}=\tau^{(p''-p')(\Sigma_{i=1}^{m_n}\rho(i)-\eta(i))}=\tau^{(p''-p')(a_n-b_n)}\leqq\tau^{(p''-p')\theta m_n}.$$

Hence, it results that

$$K_n\geqq\exp\left\{\frac{(p''-p')\theta m_n}{4}\right\}\quad\text{and}\quad\delta\geqq\frac{(p''-p')\theta}{4}.$$

The case (e) has two subcases: (e_1) $k+m_{n+1}<m_{n'+1}$ and (e_2) $m_{n'+1}\leqq k+m_{n+1}\leqq k+5m_{n+1}\leqq m_{n'+2}$, both with an odd integer n'. Then reasoning in (e_1) as in (b), and in (e_2) as in (c), after some easy computations we conclude that $\delta>\varepsilon$ and $\delta\geqq(p''-p')\theta/4$.

In conclusion, if

$$\delta=\min\left(\varepsilon,\ \frac{(p''-p')\theta}{5}\right)>0,$$

for any integers n and k we have

$$\frac{\varphi(\tau^{i+k})}{\varphi(\tau^k)\tau^{pi}}\notin[e^{-\delta m_n},e^{\delta m_n}]$$

for at least one index $i=1,2,\dots,5m_{n+1}$. Moreover, since in the construction of the sequence (n_j) we can take $n_j=hj^2$ for some big enough h, we have that $m_{n+1}=h^n(n!)^2$ and hence

$$m(n+2)=5m_{n+3}=o(e^{\sigma\delta m_n})=o(K_n^\sigma)$$

for every $\sigma>0$. This means that l^p is strongly non-equivalent to $E_{\varphi,1}$ for each $p\notin F$.

Finally, it remains to show that $\alpha_\varphi=\alpha$ and $\beta_\varphi=\beta$. Let us suppose that $p<\alpha=\alpha_{\varphi_p}$, then there exists an integer h so that

$$\frac{\varphi_p(r_h^{n+k})}{\varphi_p(r_h^n)r_h^{pk}} \leqq 1$$

for any integers $n, k \geqq 0$, and hence

$$\frac{\varphi(r_h^{n+k})}{\varphi(r_h^n)r_h^{pk}} \leqq 1.$$

Now, for $0 < \lambda \leqq 1$ and $0 < t \leqq r_h$, if we consider integers n and k verifying $r_h^n < \lambda \leqq r_h^{n-1}$ and $r_h^k \leqq t \leqq r_h^{k-1}$ $(k \geqq 2)$, we find that

$$\frac{\varphi(\lambda t)}{\varphi(\lambda)t^p} \leqq \frac{\varphi(r_h^{n+k-2})}{\varphi(r_h^n)r_h^{pk}} \leqq \frac{1}{r_h^{2p}} < \infty$$

and therefore $p \leqq \alpha_\varphi$ for every $p < \alpha$. So $\alpha \leqq \alpha_\varphi$.

Let us assume now $p < \alpha_\varphi$. There exists a constant $M > 0$ such that

$$\frac{\varphi(r_n\lambda t)}{\varphi(r_n\lambda)t^p} \leqq M$$

for $0 < \lambda \leqq 1$, $0 < t \leqq 1$, and every integer n. But

$$\lim_{n\to\infty} \frac{\varphi(r_{2n}t)}{\varphi(r_{2n})} = \varphi_p(t),$$

and so

$$\frac{\varphi_p(\lambda t)}{\varphi_p(\lambda)t^p} \leqq M.$$

Hence $p \leqq \alpha_{\varphi_p} = \alpha$ for any $p < \alpha_\varphi$, and $\alpha_\varphi \leqq \alpha$. The proof of the remaining equality $\beta_\varphi = \beta$ is similar. q.e.d.

The version of the above Theorem in the case $\alpha = 1$ takes a slightly different form because then an Orlicz sequence space l^ϕ with $\alpha_\phi = 1$ must contain a complemented copy of l^1, independently of $1 \in F$ or $1 \notin F$:

THEOREM 3. *Let $1 = \alpha \leqq \beta < \infty$ and F be a closed subset of the interval $[\alpha, \beta]$. Then there exists an Orlicz sequence space l^ϕ with indices $\alpha_\phi = 1$ and $\beta_\phi = \beta$ so that l^ϕ contains a complemented copy of l^p if and only if $p \in F \cup \{1\}$.*

PROOF. An Orlicz sequence space l^ϕ with $\alpha_\phi = 1 \leqq \beta_\phi < \infty$ contains a complemented copy of l^1 since the dual space $(l^\phi)^*$ has a subspace isomorphic to c_0 and Proposition 2.e.8 in [8]. Then, in the non-trivial case $1 = \alpha < \beta$, we can replace F by the closed set $F' = F \cup \{1\}$ and proceed in a similar way as in the above Theorem. The only change needed is to consider an Orlicz function φ_ρ, associated with the sequence ρ of 0's and 1's, having indices $1 < \alpha_{\varphi_\rho} \leqq \beta_{\varphi_\rho} = \beta$. q.e.d.

III. Complemented copies in Orlicz function spaces $L^\phi(\mu)$

We start this section extending the concept of "strongly non-equivalent function", given in ([6], [7]) for Orlicz sequence spaces, to the context of Orlicz function spaces $L^\phi(\mu)$:

DEFINITION. Let ϕ be an Orlicz function satisfying the Δ_2-condition at ∞ (resp. at 0; on $\mathbf{R}^+$). An Orlicz function ψ is called *strongly non-equivalent to* $E^\infty_{\phi,1}$ (resp. $E_{\phi,1}$; $E_{\phi,1} \cup E^\infty_{\phi,1}$) if there exist two sequences of numbers (K_n) and integers (m_n), so that for $n \to \infty$, $K_n \to \infty$ and $m_n = o(K_n^\sigma)$ for every $\sigma > 0$, and m_n-points $t_i \in (0, 1)$ such that for every $\lambda \in [\max_i t_i^{-1}, \infty)$ (resp. $\lambda \in (0, 1)$; $\lambda \in (0, \infty)$) there is at least one index i, $1 \leqq i \leqq m_n$ for which

$$\frac{\phi(\lambda t_i)}{\phi(\lambda)\psi(t_i)} \notin \left[\frac{1}{K_n}, K_n\right].$$

The following theorem generalizes a remarkable and useful result due to J. Lindenstrauss and L. Tzafriri ([6], [8] Theorem 4.b.5). Consider *weighted Orlicz sequence spaces* $l^\phi(\mu)$ for arbitrary positive scalar sequences $\mu = (\mu_n)$, i.e., the space of all sequences (x_n) verifying $\Sigma_1^\infty \phi(|x_n|/s)\mu_n < \infty$ for some $s > 0$ (cf. [2]). Clearly these spaces $l^\phi(\mu)$ have an unconditional Schauder basis in the sequence of unit vectors (e_n) when ϕ satisfies the suitable Δ_2-condition.

THEOREM 4. *Let ϕ be an Orlicz function with the Δ_2-condition at ∞ (resp. at 0) and let (μ_n) be a positive scalar sequence with $\Sigma\mu_n < \infty$ (resp. $\inf_n \mu_n > 0$). If ψ is an Orlicz function strongly non-equivalent to $E^\infty_{\phi,1}$ (resp. $E_{\phi,1}$) then l^ψ is not isomorphic to any complemented subspace of $l^\phi(\mu)$.*

PROOF. The proof is basically similar to ([6], [8] Theorem 4.b.5). We develop it putting emphasis on the necessary changes for our situation.

Let us assume that l^ψ is isomorphic to a complemented subspace of $l^\phi(\mu)$. By ([8] Propositions 1.a.12 and 1.a.9) there exists a normalized block basis $w_j = \Sigma_{i \in \sigma_j} a_i e_i$, $j \in \mathbf{N}$ of the unit vectors (e_n) in $l^\phi(\mu)$, such that (w_j) is equivalent to the unit vector basis of l^ψ, and also there is a projection $P: l^\phi(\mu) \to [w_j]$. From the conditions on the weight sequence (μ_n) it follows that we can assume, w.l.o.g., that $a_i \geqq 1$ (resp. $0 < a_i \leqq 1$) and that the functions $\psi_j(t) = \Sigma_{i \in \sigma_j} \phi(a_i t)\mu_i$ satisfy

$$|\psi_j(t) - \tilde{\psi}(t)| \leqq 1/2^j \qquad \text{for all } t \in [0, 1], \quad \text{all } j \in \mathbf{N}$$

for some Orlicz function $\tilde{\psi} \in C^\infty_{\phi,1}$ (resp. $C_{\phi,1}$) equivalent to ψ at 0.

First, the case $\Sigma\mu_n < \infty$: For convenience we will consider the function ϕ_0 defined by $\phi_0(x) = \phi(x)$ for $x \geqq 1$ and $\phi_0(x) = 0$ for $0 \leqq x < 1$. As we can suppose ϕ with the Δ_2-condition on $\mathbf{R}^+$, there is a $p > 0$ such that $\phi(st) \leqq s^p \phi(t)$ for $s \geqq 1$ and $t > 0$. Now as ψ, and hence $\tilde{\psi}$, is strongly non-equivalent to $E^\infty_{\phi,1}$, there exist $K_n > 0$ and m_n points $t_h \in (0, 1)$, $h = 1, \ldots, m_n$ with

$$\frac{m_n}{K_n^\sigma} < \min\left(\frac{1}{2^{p+1}} \| P \|^{-p}, \frac{1}{2 \cdot 3^\sigma R_0} \| P \|^{-1}\right)$$

($\sigma = 1/p$ and R_0 a positive constant), and so that for every $\lambda \geqq 1$ there exists at least one h, $1 \leqq h \leqq m_n$ such that

$$\frac{\phi_0(\lambda t_h)}{\phi_0(\lambda)\tilde{\psi}(t_h)} \notin \left[\frac{1}{K_n}, K_n\right].$$

We can assume, w.l.o.g., that $|\psi_j(t) - \tilde{\psi}(t)| < \min\{\tilde{\psi}(t_h) : 1 \leqq h \leqq m_n\}$ for all $t \in [0, 1]$.

Reasoning as in ([8] Theorem 4.b.5) we split each of the sets σ_j into $2m_n$ disjoint subsets of integers δ_j^h and η_j^h so that, for every $1 \leqq h \leqq m_n$,

$$\frac{\phi_0(a_i t_h)}{\phi_0(a_i)\tilde{\psi}(t_h)} < \frac{1}{K_n} \qquad \text{if } i \in \delta_j^h$$

and

$$\frac{\phi_0(a_i t_h)}{\phi_0(a_i)\tilde{\psi}(t_h)} > K_n \qquad \text{if } i \in \eta_j^h.$$

Then for $j \geqq 1$ and $1 \leqq h \leqq m_n$

$$K_n \tilde{\psi}(t_h) \sum_{i \in \eta_j^h} \phi_0(a_i)\mu_i \leqq \sum_{i \in \eta_j^h} \phi_0(a_i t_h)\mu_i \leqq \psi_j(t_h) \leqq 2\tilde{\psi}(t_h),$$

which implies

$$\sum_{i \in \eta_j^h} \phi_0(a_i)\mu_i \leqq \frac{2}{K_n} \quad \text{and thus} \quad \sum_{h=1}^{m_n} \sum_{i \in \eta_j^h} \phi_0(a_i)\mu_i \leqq \frac{2m_n}{K_n} \quad \text{for } j \geqq 1.$$

Now define the vectors

$$v_j = \sum_{h=1}^{m_n} \sum_{i \in \eta_j^h} a_i e_i.$$

Every function $F \in C_{\phi,1}^{\infty}$ satisfies $F(st) \leqq s^p F(t)$ for all $s \geqq 1$ and $t \geqq 1$. So, if we set

$$F_j(t) = \sum_{h=1}^{m_n} \sum_{i \in \eta_j^h} \varphi(a_i t)\mu_i,$$

then $F_j(t)/F_j(1) \in C_{\phi,1}^{\infty}$ and therefore

$$F_j(2 \| P \|) \leqq 2^P \| P \|^p F_j(1) \leqq 2^{p+1} \| P \|^p m_n / K_n \leqq 1,$$

which means that

$$\| v_j \| \leqq \frac{1}{2 \| P \|}.$$

If we write $u_j^h = \Sigma_{i \in \delta_j^h} a_i e_i$, for $1 \leqq h \leqq m_n$, $j \in \mathrm{N}$, it can be proved, in the same way as in ([8] page 151), using the "diagonal" operator, that for any set of coefficients $\{b_j\}_{j=1}^J$ the norm of the vector $\Sigma_{j=1}^J b_j w_j$ satisfies

$$\left\| \sum_{j=1}^J b_j w_j \right\| \leqq 2m_n \| P \| \left\| \sum_{j=1}^J b_j u_j^{h_j} \right\|.$$

Now, choosing an integer J so that $2M \leqq \Sigma_{j=1}^J \tilde{\psi}(t_{h_j}) \leqq 3M$ for the constant $M = K_n/3$ $(\geqq 1)$, we obtain that $M \leqq \Sigma_{j=1}^J \psi_j(t_{h_j}) \leqq 4M$. Then, by the Δ_2-condition,

$$(+) \qquad 1 \leqq M^{\sigma} \leqq \left\| \sum_{j=1}^J t_{h_j} w_j \right\| \leqq 2m_n \| P \| \left\| \sum_{j=1}^J t_{h_j} u_j^{h_j} \right\|.$$

On the other hand,

$$\sum_{j=1}^J \sum_{i \in \delta_j^{h_j}} \phi_0(a_i t_{h_j})\mu_i \leqq K_n^{-1} \sum_{j=1}^J \tilde{\psi}(t_{h_j}) \sum_{i \in \delta_j^{h_j}} \phi_0(a_i)\mu_i \leqq 1.$$

Then

$$\left\| \sum_{j=1}^J t_{h_j} u_j^{h_j} \right\| \leqq R_0$$

where R_0 is a positive constant. (If $\Sigma \mu_n \leqq \frac{1}{2}$ we can take $R_0 = 2$.) Hence for (+) we conclude that

$$\frac{m_n}{K_n^\sigma} \geqq \frac{1}{2\cdot 3^\sigma R_0} \|P\|^{-1},$$

which contradicts the choice of m_n and K_n. This ends the proof of the theorem in this first case.

The proof in the other case $\inf_n \mu_n > 0$ is analogous (except that we do not need to redefine the function ϕ). q.e.d.

REMARK. A similar result is true for the case of spaces $l^\phi(\mu)$ with arbitrary weight sequences (μ_n). Namely, if a function ψ is strongly non-equivalent to $E_{\phi,1}^\infty \cup E_{\phi,1}$ then l^ψ is not isomorphic to any complemented subspace of $l^\phi(\mu)$.

PROPOSITION 5. *If $L^\phi(\mu)$ is a reflexive Orlicz space over a finite (or σ-finite) measure space (Ω, Σ, μ), then $L^\phi(\mu)$ contains a complemented copy of l^p for $p \neq 2$ if and only if l^p is isomorphic to a complemented subspace of a weighted Orlicz sequence space $l^\phi(\mu)$ for $\mu_n = \mu(A_n)$ of a disjoint sequence (A_n) in Σ.*

PROOF. The "if" implication is obvious since the sequence space $l^\phi(\mu)$ is canonically embedded into $L^\phi(\mu)$ as a complemented subspace.

Let us suppose now that $L^\phi(\mu)$ contains a complemented copy of l^p for $p > 2$. Reasoning in the same manner as in the proof of Proposition 4 of [1] (also [12] for μ σ-finite), i.e. using ([9] Proposition 1.c.8), we get that there exists a disjointly supported sequence $(g_n)_1^\infty$ in $L^\phi(\mu)$, so that its span $[g_n]$ is isomorphic to l^p and complemented in $L^\phi(\mu)$. Now, by the density of the step functions in $L^\phi(\mu)$, for each $n \in \mathrm{N}$ there are mutually disjoint sets $B_{k,n} \subset \mathrm{supp}(g_n) = A_n$ and real numbers $(a_{k,n})_{k=1}^{m_n}$ such that $h_n = \Sigma_{k=1}^{m_n} a_{k,n}\chi_{B_{k,n}}$ verifies $\| h_n - g_n \| < 1/2^n$. Hence, by a standard perturbation result ([8] Proposition 1.a.9), the span $[h_n]$ is isomorphic to l^p. Finally, the subspace $[(\chi_{B_{k,n}})]_{k,n}$, which is isomorphic to the weighted space $l^\phi(\mu_{k,n})$ for $\mu_{k,n} = \mu(B_{k,n})$, contains a complemented subspace isomorphic to l^p.

In the other case $1 < p < 2$, the result is now easily proved by duality (see, e.g., [12] Theorem 14). q.e.d.

A direct consequence of Theorem 4 and Proposition 5 is the following Corollary that gives us a version for function spaces $L^\phi(\mu)$ of the J. Lindenstrauss and L. Tzafriri result for Orlicz sequence spaces l^ϕ ([6], [8] Theorem 4.b.5):

COROLLARY 6. *Let $L^\phi(\mu)$ be a reflexive Orlicz function space over a finite (resp. σ-finite) measure space (Ω, μ). If the function t^p, for $p \neq 2$, is strongly* *non-equivalent to* $E^{\infty}_{\phi,1}$ *(resp.* $E^{\infty}_{\phi,1} \cup E_{\phi,1}$*) then the space* l^p *is not isomorphic to a complemented subspace of* $L^{\phi}(\mu)$.

We are now ready to study the *inverse problem* in the context of Orlicz function spaces $L^{\phi}(\mu)$:

THEOREM 7. *Let* $1 < \alpha \leqq \beta < \infty$, *F be a closed subset of* $[\alpha, \beta]$, *and* (Ω, μ) *be a non-purely atomic finite measure space. Then there exists an Orlicz function space* $L^{\phi}(\mu)$ *with indices* $\alpha^{\infty}_{\phi} = \alpha$ *and* $\beta^{\infty}_{\phi} = \beta$ *such that* $L^{\phi}(\mu)$ *contains a complemented copy of* l^p *if and only if* $p \in F \cup \{2\}$.

PROOF. We can assume $F \neq \varnothing$. The case $F = \varnothing$ was solved in [1] and also below in Corollary 10.

First, let us prove the existence of an Orlicz function ϕ with indices $\alpha^{\infty}_{\phi} = \alpha$ and $\beta^{\infty}_{\phi} = \beta$, so that t^p is equivalent at 0 to a function of E^{∞}_{ϕ} if $p \in F$, and t^p is strongly non-equivalent to $E^{\infty}_{\phi,1}$ if $p \notin F$. Indeed, let us define the symmetric function $\tilde{\varphi}(x) = 1/\varphi(1/x)$ of the function φ considered in Theorem 2, i.e.,

$$\tilde{\varphi}(x) = \begin{cases} \left(\dfrac{x}{e^{m_n}}\right)^{p_n} \dfrac{1}{\varphi(e^{-m_n})} & \text{if } e^{m_n} \leqq x \leqq e^{m_{n+1}} \text{ and } n \text{ odd} \\[2ex] \dfrac{1}{\varphi_p\left(\dfrac{e^{m_n}}{x}\right)\varphi(e^{-m_n})} & \text{if } e^{m_n} \leqq x \leqq e^{m_{n+1}} \text{ and } n \text{ even} \end{cases}$$

where the sequences (p_n) and (m_n) and the function φ_p are as in Theorem 2. From the equality $E^{\infty}_{\tilde{\varphi}} = (\tilde{E}_{\varphi})$ we obtain that t^p is equivalent at 0 to a function of $E^{\infty}_{\tilde{\varphi}}$ if $p \in F$. Moreover $\alpha^{\infty}_{\tilde{\varphi}} = \alpha$ and $\beta^{\infty}_{\tilde{\varphi}} = \beta$.

Now, to show that t^p is strongly non-equivalent to $E^{\infty}_{\tilde{\varphi},1}$ if $p \notin F$, we proceed in the same way as in Theorem 2. Let $\varepsilon > 0$ be so that $(p - 3\varepsilon, p + 3\varepsilon) \cap F = \varnothing$. For each odd integer n put $m(n) = 5m_{n+1}$ and assume there exists an integer $k \geqq 5m_{n+1}$ verifying

$$K_n^{-1}\tau^{-pi} \leqq \frac{\tilde{\varphi}(\tau^k\tau^{-i})}{\tilde{\varphi}(\tau^k)} \leqq K_n\tau^{-pi}$$

for $i = 1, 2, \ldots, 5m_{n+1}$; $\tau = e$ and $K_n = e^{\delta m_n}$ $(\delta > 0)$. Let $1 \leqq j \leqq 4m_{n+1}$; by using the above inequality with $i = j$ and $i = j + m_n$ it results that

$$K_n^{-2}\tau^{-pm_n} \leqq \frac{\tilde{\varphi}(\tau^{k-j-m_n})}{\tilde{\varphi}(\tau^{k-j})} \leqq K_n^2\tau^{-pm_n}$$

for $k \geqq 5m_{n+1}$.

Now, for each one of the following cases we have an appropriate index $1 \leqq j \leqq 4m_{n+1}$, which leads to a contradiction (we omit the details). Namely,

(a) The case $k < 5m_{n+1}$ is now excluded.

(b) For $m_{n'} \leqq k - 4m_{n+1} \leqq k \leqq m_{n'+1}$, with $n' \geqq n$ and odd, take $j = 1$.

(c) For $m_{n'} \leqq k - 4m_{n+1} \leqq k \leqq m_{n'+1}$, with $n' \geqq n$ and even, take j_1, j_2 as in the case (c) of Theorem 2.

(d) For $m_{n'} \leqq k - 4m_{n+1} \leqq m_{n'+1} < k$ with $n' \geqq n$ and odd, take $j = k - m_{n'+1}$.

(*e*) For $m_{n'} \leqq k - 4m_{n+1} \leqq m_{n'+1} < k$ with $n' \geqq n$ and even, we have two subcases:

(e_1) If $m_{n'+1} \leqq k - m_{n+1} \leqq k < m_{n'+2}$, take $j = 1$.

(e_2) If $m_{n'} \leqq k - 5m_{n+1} \leqq k - m_{n+1} < m_{n'+1}$, we consider j_1, j_2 as in the respective (e_2) subcase in the proof of Theorem 2.

In conclusion, t^p is strongly non-equivalent to a function of $E^{\infty}_{\tilde{\varphi},1}$.

Finally, let us consider the Orlicz space $L^{\phi}(\mu)$ where ϕ is a convex function equivalent to $\tilde{\varphi}$ at ∞ (f.i. $\phi(t) = \int_1^t (\tilde{\varphi}(x)/x)dx$). Thus $\alpha_\phi^\infty = \alpha$ and $\beta_\phi^\infty = \beta$, and the reflexive Orlicz space $L^\phi(\mu)$ has a complemented subspace (the Rademacher functions span) isomorphic to l^2 (cf. [9]). Also $L^\phi(\mu)$ contains a complemented copy of l^p for each $p \in F$ since $t^p \in E_\phi^\infty$ and ([7] Proposition 4.4). The converse is immediate using Corollary 6. q.e.d.

Remark 1. In the case $1 = \alpha < \beta < \infty$ we could give some partial results, but not as complete as for Orlicz sequence spaces l^ϕ (Theorem 3).

Remark 2. Let us denote by Q_ϕ^∞ (resp. Q_ϕ) the set of those $p \geqq 1$ for which the Orlicz function space $L^\phi(\mu)$ (resp. Orlicz sequence space l^ϕ) contains a complemented copy of l^p. Theorems 7 and 2 give rise to the question of whether the sets Q_ϕ^∞ and Q_ϕ are always closed for every Orlicz function ϕ.

The above results will be considered now in the context of the class of minimal Orlicz functions studied in [6], [7], [8] and [1]. Firstly, let us introduce the following

Definition. An Orlicz function ϕ with the Δ_2-condition is called *distinctive* (resp. at ∞; at 0) if for any positive sequence (μ_n) (resp. with $\Sigma\mu_n < \infty$;

$\inf_n \mu_n > 0$) the weighted sequence space $l^\phi(\mu)$ is isomorphic to the Orlicz sequence space l^ϕ.

In other words a function ϕ is distinctive at 0 if every block basis with constant coefficients of the canonical basis (e_n) of l^ϕ spans a subspace which is isomorphic to l^ϕ itself.

The functions t^p are trivial examples of distinctive functions, and the complementary function $\hat{\phi}$ of a distinctive function ϕ is also a distinctive function, since

$$l^{\hat{\phi}}(\mu) \simeq (l^\phi(\mu))^* \simeq (l^\phi)^* \simeq l^{\hat{\phi}}$$

for any arbitrary weight sequence (μ_n).

Proposition 5 takes, for this class of distinctive functions, the following nice form:

PROPOSITION 8. *Let $L^\phi(\mu)$ be a distinctive reflexive Orlicz space over a finite (or σ-finite) measure space not reduced to a finite number of atoms. Then $L^\phi(\mu)$ contains a complemented copy of l^p for $p \neq 2$ if and only if l^ϕ contains a complemented copy of l^p.*

Inside the class of the distinctive functions at 0 are the important class of the minimal functions introduced by J. Lindenstrauss and L. Tzafriri ([6], [7], see also [8]). This follows from Proposition 4.b.7. in [8].

More generally, it occurs that the general minimal functions studied in [1], [2] are distinctive functions: Recall that ϕ is called *minimal* (at ∞) if for $\psi \in E^\infty_{\phi,1}$, as a subset of the space $C(0, \infty)$ endowed with the compact-open topology, $E^\infty_{\psi,1} = E^\infty_{\phi,1}$.

PROPOSITION 9. *If ϕ is a minimal function then ϕ is a distinctive function.*

PROOF. Let $\mu = (\mu_n)$ be an arbitrary weight sequence and positive scalars $r_n > 0$ so that $\mu_n = 1/\phi(r_n)$. Then the functions $(\phi(r_n \cdot)/\phi(r_n))$ belong to $E^\infty_{\phi,1} \cup E_{\phi,1}$. If (e_n) denotes the canonical basis of l^ϕ, the sequence $(f_n) = (r_n e_n)$ is a basis for $l^\phi(\mu)$. By the minimality of the function ϕ ([1] Proposition 1) we have that $E^\infty_{\phi,1} = E_{\phi,1} = E_\phi$, and so we can take a sequence (s_n) converging to 0 such that

$$\left| \frac{\phi(r_n t)}{\phi(r_n)} - \frac{\phi(s_n t)}{\phi(s_n)} \right| \leq \frac{1}{2^n}$$

for all $t \in [0, 1]$. This implies that for $w_n = 1/\phi(s_n)$, the basis $(g_n) = (s_n e_n)$ of $l^\phi(w)$ and (f_n) are equivalent. Hence the space $l^\phi(\mu)$ is isomorphic to $l^\phi(w)$ with $w_n \to \infty$, which by Proposition 4.b.7 of ([8]) is isomorphic to l^ϕ. q.e.d.

The above results allow us to answer a question in [1] (Remark, page 360) to show the main Theorem in [1] is also true for the case $\alpha_\phi^\infty < 2 < \beta_\phi^\infty$:

COROLLARY 10. *Given $1 < \alpha \leqq \beta < \infty$ arbitrary, there exists a minimal Orlicz function space $L^\phi(\mu)$ over a finite (or σ-finite) measure space with indices $\alpha_\phi^\infty = \alpha$ and $\beta_\phi^\infty = \beta$ and which contains no complemented subspace isomorphic to l^p for any $p \neq 2$.*

PROOF. Let us consider the minimal Orlicz function ϕ defined by J. Lindenstrauss and L. Tzafriri in [6], [8] (Examples 4.c.6 or 4.c.7) with indices $\alpha_\phi = \alpha$ and $\beta_\phi = \beta$. Hence the Orlicz sequence space l^ϕ does not have any complemented copy of l^p for $p \geqq 1$. Then the restriction of the function ϕ to [0, 1] can be extended to the whole $[0, \infty)$ defining a general minimal function ([1] page 357) denoted also by ϕ. Thus, the associated indices satisfy $\alpha_\phi^\infty = \alpha_\phi = \alpha$ and $\beta_\phi^\infty = \beta_\phi = \beta$ ([2]), and by appeal to Propositions 9 and 8 this finishes the proof. q.e.d.

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